

Test Assignment #5

① We want to sketch the graph of $y = x^2 e^x$.

(i) Domain is $x \in (-\infty, \infty)$

(ii) To find the x -intercepts, we assign $y = 0$.

$$\text{so } x^2 e^x = 0$$

$$\Rightarrow x^2 = 0 \text{ since } e^x \neq 0$$

$$\Rightarrow x = 0 \quad \therefore x\text{-intercept occurs at } (0, 0)$$

To find the y -intercept, we assign $x = 0$

$$\text{so } y = 0^2 e^0 = 0 \quad \therefore y\text{-intercept occurs at } (0, 0)$$

(iii) $\lim_{x \rightarrow -\infty} (x^2 e^x) = 0$ indicates a horizontal asymptote $y = 0$ to the left of the graph

$\lim_{x \rightarrow \infty} (x^2 e^x) = \infty$ indicates no horizontal asymptote to the right of the graph.

Since the domain is $x \in (-\infty, \infty)$, there are no vertical asymptotes.

$$(iv) \quad \frac{dy}{dx} = x(x+2)e^x$$

Critical points occur when

$$(1) \quad \frac{dy}{dx} = 0 \Rightarrow x(x+2)e^x = 0$$

$$\Rightarrow x = 0 \text{ or } x+2 = 0 \text{ but } e^x \neq 0$$

$$\Rightarrow x = 0 \text{ or } x = -2$$

Acceptable since they are on the domain

(2) $\frac{dy}{dx}$ does not exist.

But $\frac{dy}{dx} = x(x+2)e^x$ exists everywhere on the domain

so this criteria does not yield any critical points.

	$x < -2$	$-2 < x < 0$	$x > 0$
Sign of x	-ve	-ve	+ve
Sign of $x+2$	-ve	+ve	+ve
Sign of e^x	+ve	+ve	+ve
Sign of $\frac{dy}{dx} = x(x+2)e^x$	+ve	-ve	+ve
Orientation of $y = x^2 e^x$	/ Inc.	Rel. Max.	Rel. Min.

When $x = -2$, then $y = (-2)^2 e^{-2} = \frac{4}{e^2} \approx 0.541341$

When $x = 0$, then $y = 0^2 e^0 = 0$

$\therefore$ Rel max at $(-2, 0.541341)$

Rel min at $(0, 0)$

(v) $\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$

Inflection points may occur when

(1) $\frac{d^2y}{dx^2} = 0 \Rightarrow (x^2 + 4x + 2)e^x = 0$

$\Rightarrow x^2 + 4x + 2 = 0$ but $e^x \neq 0$

$\Rightarrow x = \frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2(1)} = \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2}$

$\Rightarrow x = -2 \pm \sqrt{2} \Rightarrow x = -2 - \sqrt{2}$ or $x = -2 + \sqrt{2}$

$\Rightarrow x \approx -3.414214$ or $x \approx -0.585786$

Acceptable since they are on the domain

(2) $\frac{d^2y}{dx^2}$ does not exist

But $\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$ exists everywhere on the

domain so this criteria does not yield any possible inflection points.

$$x < -3.414 \quad -3.414 < x < -0.586 \quad x > -0.586 \rightarrow x$$

Sign of $x^2 + 4x + 2$

Sign of e^x

Sign of $\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$

Concavity of $y = x^2 e^x$

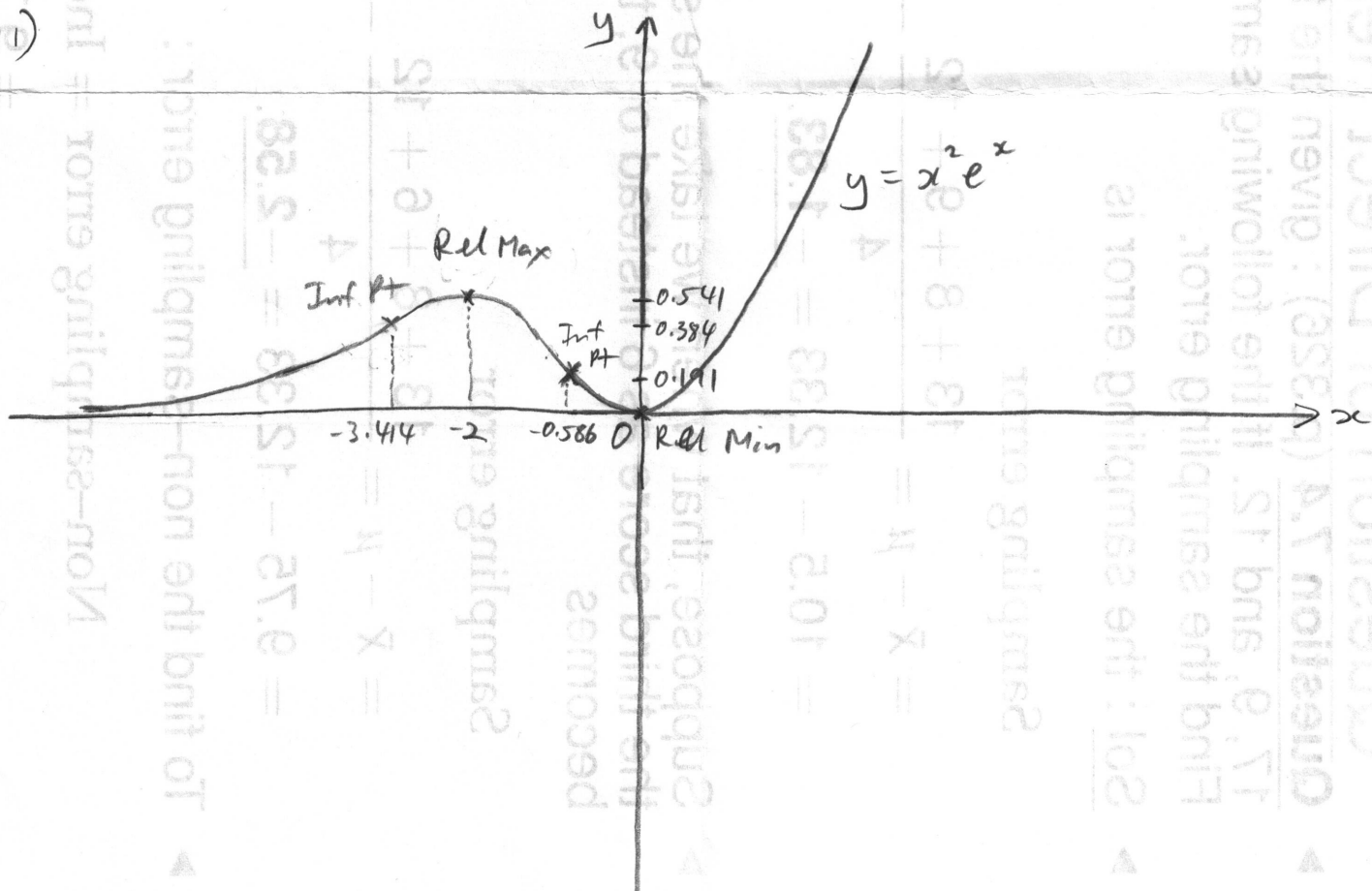
+ve	-ve	+ve
+ve	+ve	+ve
+ve	-ve	+ve
∪ conc. up	∩ conc. down	∪ conc. up
Inf. Pt		Inf. Pt

When $x = -2 - \sqrt{2} \approx -3.414$, then $y = (-3.414)^2 e^{-3.414} \approx 0.383537$

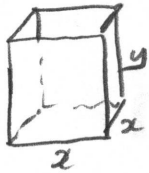
When $x = -2 + \sqrt{2} \approx -0.586$, then $y = (-0.586)^2 e^{-0.586} \approx 0.191018$

∴ Inflection points occur at $(-3.414, 0.384)$
and at $(-0.586, 0.191)$

(v)



(2)



Let the dimensions of the box be x m, x m and y m as shown.

We are told that the volume is 6 m^3

$$\Rightarrow x^2 y = 6 \text{ m}^3$$

The square base has area x^2

so the base costs $\$3x^2$

Each side has area xy and there are 4 sides

so the sides cost $\$2(4)(xy) = \$8xy$

Let C be the total costs of the box

$$\Rightarrow C = 3x^2 + 8xy$$

$$\text{From } x^2 y = 6 \Rightarrow y = \frac{6}{x^2}$$

Substitute this into C ,

$$C = 3x^2 + 8x\left(\frac{6}{x^2}\right)$$

$$\Rightarrow C = 3x^2 + \frac{48}{x} ; x > 0$$

To find the critical points, we need

$$\frac{dC}{dx} = 6x - \frac{48}{x^2} = \frac{6x^3}{x^2} - \frac{48}{x^2} = \frac{6x^3 - 48}{x^2}$$

Critical points occur when

$$(1) \frac{dC}{dx} = 0 \Rightarrow \frac{6x^3 - 48}{x^2} = 0$$

$$\Rightarrow 6x^3 - 48 = 0$$

$$\Rightarrow 6x^3 = 48$$

$$\Rightarrow x^3 = \frac{48}{6} = 8$$

$$\Rightarrow x = \sqrt[3]{8} = 2 \text{ m}$$

Acceptable since it is on the domain.

$$(2) \quad \frac{dC}{dx} \text{ does not exist} \Rightarrow \frac{6x^3 - 48}{x^2} \text{ does not exist}$$

$$\Rightarrow x^2 = 0$$

$$\Rightarrow x = 0$$

Not acceptable since it is not on the domain.

$\therefore C = 3x^2 + \frac{48}{x}$ has only one crit. pt. on its domain
and the Unique Extremum Theorem applies.

$$\text{From } \frac{dC}{dx} = 6x - \frac{48}{x^2}$$

$$\Rightarrow \frac{d^2C}{dx^2} = 6 - 48\left(-\frac{2x}{(x^2)^2}\right) = 6 + \frac{96}{x^3}$$

$$\text{when } x = 2, \text{ then } \frac{d^2C}{dx^2} = 6 + \frac{96}{2^3} > 0$$

indicating a relative minimum at $x = 2$.

Since it is the only critical point on the domain,
the Unique Extremum Theorem applies and it is
also the Absolute Minimum point.

$$\text{when } x = 2 \text{ m, then } y = \frac{6}{2^2} = \frac{6}{4} = 1.5 \text{ m}$$

$\therefore$ The box with the lowest cost has dimensions
2 m, 2 m and 1.5 m.

(3) Given $p = \sqrt{2500 - 50q}$ where $0 \leq q \leq 50$

$$(i) \quad p^2 = 2500 - 50q$$

$$\Rightarrow 50q = 2500 - p^2$$

$$\Rightarrow q = \frac{2500 - p^2}{50} = \frac{2500}{50} - \frac{p^2}{50}$$

$$\Rightarrow q = 50 - 0.02p^2$$

$$\text{When } q=0, \text{ then } p = \sqrt{2500 - 50(0)} = \sqrt{2500} = 50$$

$$\text{When } q=50, \text{ then } p = \sqrt{2500 - 50(50)} = \sqrt{0} = 0$$

So the domain of $q = 50 - 0.02p^2$ is $0 \leq p \leq 50$

$$(ii) \quad \eta = -\frac{p}{q} \cdot \frac{dq}{dp} \quad \text{We need } \frac{dq}{dp} = -0.04p$$

$$\text{So } \eta = -\frac{p}{50 - 0.02p^2} (-0.04p) =$$

$$\Rightarrow \eta = \frac{0.04p^2}{50 - 0.02p^2}$$

$$(iii) \quad \text{When } p = \$30, \text{ then } \eta = \frac{0.04(30^2)}{50 - 0.02(30^2)} = \frac{36}{32} = 1.125$$

$\therefore$ The demand is elastic.

(iv) Revenue is maximised when $\eta = 1$

$$\Rightarrow \frac{0.04p^2}{50 - 0.02p^2} = 1$$

$$\Rightarrow 0.04p^2 = 50 - 0.02p^2$$

$$\Rightarrow 0.04p^2 + 0.02p^2 = 0.06p^2 = 50$$

$$\Rightarrow p^2 = \frac{50}{0.06} = \frac{2500}{3}$$

$$\Rightarrow p = \sqrt{\frac{2500}{3}} \approx \$28.87. \text{ We reject } p < 0.$$

$$\text{And } q = 50 - 0.02\left(\sqrt{\frac{2500}{3}}\right)^2 = 50 - 0.02\left(\frac{2500}{3}\right) = 50 - \frac{50}{3}$$
$$= \frac{100}{3} \approx 33 \frac{1}{3} \approx 33 \text{ units}$$

$$\therefore R_{\max} = p_{\max} q_{\max} = \sqrt{\frac{2500}{3}} \cdot \frac{100}{3} \approx \$962.25$$